

AXIOMATIC DESCRIPTION OF LEFSCHETZ TYPE EQUIVARIANT HOMOTOPY INVARIANTS

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ABSTRACT. We describe generators of universal Lefschetz groups consisting of self-maps of equivariant 1-spheres. This allows to formulate a normalization axiom which, together with the usual axioms, determines an equivariant Lefschetz number uniquely. We apply this technique to identify the fixed orbit detecting equivariant Lefschetz number of Chorny with the fixed orbit index of Dzedzej.

1. INTRODUCTION

The Lefschetz number is a homotopy invariant of a self-map $f : X \rightarrow X$ of a finite CW complex intimately related to the fixed points of the map f . Arkowitz and Brown have shown in [1] that the Lefschetz number is uniquely characterized by four axioms, which are quite natural to postulate.

Things are a bit trickier in the equivariant world. Laitinen and Lück [6] have constructed a universal Lefschetz group and a universal Lefschetz number, where universality means that every equivariant homotopy invariant with certain properties is the homomorphic image of the universal invariant. There are still several useful instances of equivariant Lefschetz numbers, and it is desirable to be able to tell when two such invariants are equal, i.e. to determine the homomorphism which specifies them.

In this paper we are going to give a characterization of the universal Lefschetz group for any topological group G in terms of generators. The generators consist of equivariant homotopy classes of self maps of finite wedge sums of equivariant 1-spheres. By introducing a linearity axiom in the definition of an equivariant Lefschetz number, we obtain a universal linear Lefschetz group and a universal linear equivariant Lefschetz number. This is the Lefschetz invariant of Laitinen and Lück. For the universal linear Lefschetz group, we obtain a simpler set of generators, which turns out to be a basis under mild hypotheses on G . This last result was already imminent in [6].

Formulated in the Axiomatic language in the spirit of Arkowitz and Brown, the characterization implies that any equivariant Lefschetz number is uniquely determined by its values on self maps of finite wedge sums of equivariant 1-spheres. This extends a result of Gonçalves and Weber [4], who have identified yet another equivariant Lefschetz number of Lück and Rosenberg [7] for cocompact actions as the unique equivariant Lefschetz invariant having as values on finite wedge sums of equivariant 1-spheres certain linear combinations of incidence numbers.

In Section 2, we develop the universal Lefschetz group and the universal linear Lefschetz group, construct the universal invariants, and describe generators and

bases of the universal groups. The main result is Theorem 2.6. In Corollary 2.7, we reformulate the result as a set of axioms for equivariant Lefschetz numbers.

In Section 3, we develop the theory of equivariant cellular homology to be able to define an equivariant Lefschetz number later. This chapter is based on [2] and [11] and nothing new is proven here. At some points the representation may be a bit more simple than in the references, but on the other hand, we have also skipped some of the proofs.

Section 4 reviews the theory of the Hattori-Stallings trace of endomorphisms of finitely generated projective modules and proves some of the more unknown properties of this map.

In Section 5 and Section 6, we use the axioms to compare two constructions of an equivariant Lefschetz number which, in contrast to Lück and Rosenbergs invariant, detects fixed orbits of G -maps rather than fixed points. These constructions are a homological construction, extracted from [2], and an analytical construction done by Dzedzej in [3]. It turns out that these two invariants are equal whenever both are defined. In many instances, the analytical Lefschetz number is much easier to calculate than the homological one.

We close in Section 7 with the obligatory Lefschetz fixed orbit theorem and an example that shows that the formulation is, in some sense, optimal.

2. THE UNIVERSAL LEFSCHETZ NUMBER

In the following we will work in the category of pointed G -CW complexes, where G is any topological group. G will be more specific later on. A quick definition of the notion of G -CW complexes is given at the beginning of Section 3. We will not refer to the fact that complexes and maps are pointed. So by a G -map $f : X \rightarrow X$ of a G -CW complex, we really mean a pointed G -map of a pointed G -CW complex with G -fixed base point. Similarly, equivariant homotopies are required to fix base points.

Definition 1. An equivariant Lefschetz number is a pair (ℓ_G, B) , where B is an abelian group and ℓ_G is an assignment, assigning to any equivariant self map f of a finite G -CW complex X an element $\ell_G(f)$ of B . This is required to fulfill:

- (1) (Homotopy Invariance): If $k : X \rightarrow Y$ is a G -homotopy equivalence of finite G -CW complexes, $f : X \rightarrow X$, $h : Y \rightarrow Y$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow k & & \downarrow k \\ Y & \xrightarrow{h} & Y \end{array}$$

is G -homotopy commutative, then $\ell_G(f) = \ell_G(h)$.

- (2) (Cofibration): If $A \subseteq X$ is an invariant subcomplex and there is a diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ f|_A \downarrow & & \downarrow f & & \downarrow \hat{f} \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array}$$

then

$$\ell_G(f) = \ell_G(f|_A) + \ell_G(\hat{f}).$$

- (3) (Commutativity): If X, Y are G -CW complexes and $f : X \rightarrow Y, h : Y \rightarrow X$ are G -maps, then

$$\ell_G(h \circ f) = \ell_G(f \circ h).$$

A universal equivariant Lefschetz number is a Lefschetz number (L_G, U) such that, whenever (ℓ_G, B) is an equivariant Lefschetz number, there is a unique homomorphism $\varphi : U \rightarrow B$ such that $\ell_G(f) = \varphi(L_G(f))$ for any self map $f : X \rightarrow X$ of a finite G -CW complex.

The following purely formal result shows that a universal equivariant Lefschetz number exists.

Proposition 2.1. *There exists a universal Lefschetz number $(L_G, \mathbb{U}_G L)$.*

Proof. Consider the set of pairs $[f, X]$ of equivalence classes of pairs (f, X) , where (f, X) is equivalent to (h, Y) , if there is a G -homotopy equivalence $k : X \rightarrow Y$, making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow k & & \downarrow k \\ Y & \xrightarrow{h} & Y \end{array}$$

G -homotopy commutative. Let U be the free abelian group on these equivalence classes and S be the subgroup generated by elements of one of the following forms.

(1)

$$[f, X] - [f|_A, A] - [\hat{f}, X/A],$$

where $A \subseteq X$ is an invariant subcomplex, $f(A) \subseteq A$ and \hat{f} is the map induced in the quotient.

(2)

$$[f \circ h, X] - [h \circ f, Y],$$

where $h : X \rightarrow Y$ and $f : Y \rightarrow X$ are G -maps.

Denote the quotient group U/S by $\mathbb{U}_G L$. For a self-map $f : X \rightarrow X$, let $L_G(f) \in \mathbb{U}_G L$ be the class of $[f, X]$. We claim that this is a universal Lefschetz invariant. Indeed, if (ℓ_G, B) is any Lefschetz invariant, we can define a map $\varphi : \mathbb{U}_G L \rightarrow B$ by $[f, X] \mapsto \ell_G(f)$ and extend linearly. This is well-defined because ℓ_G is an equivariant Lefschetz number and obviously, φ is a homomorphism with $\varphi(L_G(f)) = \ell_G(f)$. \square

Before we describe the universal Lefschetz group more explicitly, we derive some properties of equivariant Lefschetz numbers in general.

Proposition 2.2. *Let (ℓ_G, B) be an equivariant Lefschetz number. Then the following holds.*

- (i) *For the one point space $*$ and the unique map $* : * \rightarrow *$, $\ell_G(*) = 0$.*
- (ii) *If $f : X \rightarrow X$ is equivariantly homotopic to the constant map, then $\ell_G(f) = 0$.*
- (iii) *For a G -CW complex X , its ordinary suspension $\Sigma X = X \wedge \mathbb{S}^1$ and $f : X \rightarrow X$,*

$$\ell_G(f) = -\ell_G(\Sigma f).$$

Proof. (i) Clearly for the sequence $* \xrightarrow{*} * \xrightarrow{*} *$, the cofibration property of the equivariant Lefschetz number implies

$$\ell_G(*) = \ell_G(*) + \ell_G(*)$$

and consequently $\ell_G(*) = 0$.

- (ii) Let $c : X \rightarrow X$ be the constant map. c factors as $c = i \circ p$, where $p : X \rightarrow *$ is the projection, $i : * \rightarrow X$ the inclusion of the base point. By the commutativity property and (i),

$$\ell_G(c) = \ell_G(i \circ p) = \ell_G(p \circ i) = \ell_G(*) = 0.$$

- (iii) Let CX be the reduced cone over X , then $\Sigma X \cong CX/X$ and the map $Cf : CX \rightarrow CX$ preserves $X \subseteq CX$. Furthermore, the map Cf induces on the quotient is just Σf . Hence we obtain

$$\ell_G(Cf) = \ell_G(f) + \ell_G(\Sigma f).$$

Since CX is G -contractible, Cf is equivariantly homotopic to the constant map and therefore $\ell_G(Cf) = 0$, which yields the result. \square

The next lemma is crucial for several inductive proofs on the dimension of CW complexes.

Lemma 2.3. *Let $X = \bigvee G/H_{i+} \wedge \mathbb{S}^{n-1}$ be a finite wedge of equivariant $(n-1)$ -spheres, $f : X \rightarrow X$ a G -map. Then f is equivariantly homotopic to the $(n-2)$ -nd ordinary suspension of a G -map of the complex*

$$Y = \bigvee G/H_{i+} \wedge \mathbb{S}^1,$$

provided $n \geq 2$.

Proof. We want to show that the suspension homomorphism

$$\Sigma : [G/K_+ \wedge \mathbb{S}^{n-1}, X]_G \rightarrow [G/K_+ \wedge \mathbb{S}^n, \Sigma X]_G$$

is an epimorphism under the condition that X is a wedge sum of equivariant $(n-1)$ -spheres. We have the usual adjunction isomorphism

$$[G/K_+ \wedge \mathbb{S}^{n-1}, X]_G \cong [\mathbb{S}^{n-1}, X^K].$$

Hence, it suffices to show that

$$\Sigma : [\mathbb{S}^{n-1}, X^K] \rightarrow [\mathbb{S}^n, \Sigma X^K]$$

is an epimorphism. By the Freudenthal suspension theorem, if X^K is m -connected, then

$$\Sigma : \pi_r(X^K) \rightarrow \pi_{r+1}(\Sigma X^K)$$

is an epimorphism for $r \leq 2m+1$. We have

$$\begin{aligned} X^K &= \{([g], x) \mid ([kg], x) = ([g], x) \ \forall k \in K\} \cup \{*\} \\ &= \{([g], x) \mid ([g], x) \in G/H_i \times \mathbb{S}^{n-1}, \ g^{-1}Kg \subseteq H_i\} \cup \{*\} \\ &= \left(\bigvee_{i \in I} N(K, H_i)_+ \right) \wedge \mathbb{S}^{n-1}, \end{aligned}$$

where $N(K, H) = \{g \in G \mid g^{-1}Kg \subseteq H\}$. Since \mathbb{S}^{n-1} is $(n-2)$ -connected and X^K is the smash product of \mathbb{S}^{n-1} with a -1 -connected space, X^K is itself $(n-2)$ -connected. So the Freudenthal theorem yields that suspension is an epimorphism, provided $r \leq 2(n-2) + 1 = 2n-3$. Letting $r = n-1$, the condition reduces to $0 \leq n-2$, i.e. $n \geq 2$. \square

In the present generality, equivariant Lefschetz numbers are not well-suited for applications. For example, there is in general no way to give a connection between an equivariant Lefschetz number and the fixed points or orbits of a map. Therefore it is customary to assume further axioms. By universality, the resulting invariants will be homomorphic images of the universal Lefschetz number.

Laitinen and Lück in [6] have introduced the following Axiom.

4a. (Linearity): If X is a G -CW complex and $f, h : \Sigma X \rightarrow \Sigma X$, then

$$\ell_G(f + h) = \ell_G(f) + \ell_G(h),$$

where $f + h$ is a representant of the sum of the homotopy classes of f and h , coming from the structure of ΣX as an H -cogroup.

Arkowitz and Brown in [1] gave an additivity Axiom, though without an acting group G . An equivariant version appears in [4]. It is apparent that additivity must hold for any Lefschetz number that is related to fixed points or orbits of a map.

4b. (Wedge of Circles): Whenever $X = \bigvee G/H_{i+} \wedge \mathbb{S}^1$ is a finite wedge of equivariant 1-spheres and $f : X \rightarrow X$ a G -map, then

$$\ell_G(f) = \sum \ell_G(f_i),$$

where $f_i = \pi_i \circ f \circ \iota_i$, with $\iota_i : G/H_{i+} \wedge \mathbb{S}^1 \rightarrow X$ the inclusion of the i -th sphere, $\pi_i : X \rightarrow G/H_{i+} \wedge \mathbb{S}^1$ the projection onto it.

The wedge of circles axiom ensures that an equivariant Lefschetz number is additive in general.

Proposition 2.4. *Let (ℓ_G, B) be an equivariant Lefschetz number satisfying the wedge of circles axiom and let $X = \bigvee X_i$ be a finite wedge of G -CW complexes, $f : X \rightarrow X$ a G -map. Then*

$$\ell_G(f) = \sum \ell_G(f_i),$$

where f_i is the obvious adjustment of the map f_i from Axiom 4b.

Proof. We begin by proving that the assertion holds for wedge sums of equivariant n -spheres. Hence, let

$$X = \bigvee G/H_{i+} \wedge \mathbb{S}^n, f : X \rightarrow X.$$

By Lemma 2.3, f is equivariantly homotopic to the repeated suspension of a G -map $h : Y \rightarrow Y$, where

$$Y = \bigvee G/H_{i+} \wedge \mathbb{S}^1.$$

By the wedge of circles Axiom, $\ell_G(h) = \sum \ell_G(h_i)$. By homotopy invariance and the obvious equality $\pi_i \circ \Sigma h \circ \iota_i = \Sigma(\pi_i \circ h \circ \iota_i)$, it follows that

$$\begin{aligned} \ell_G(f_i) &= \ell_G(\pi_i \circ f \circ \iota_i) \\ &= \ell_G(\pi_i \circ \Sigma^{n-1} h \circ \iota_i) \\ &= \ell_G(\Sigma^{n-1} h_i) \\ &= (-1)^{n-1} \ell_G(h_i). \end{aligned}$$

Therefore,

$$\begin{aligned}
\ell_G(f) &= \ell_G(\Sigma^{n-1}h) \\
&= (-1)^{n-1}\ell_G(h) \\
&= (-1)^{n-1}\sum \ell_G(h_i) \\
&= (-1)^{n-1}\sum (-1)^{n-1}\ell_G(f_i) \\
&= \sum \ell_G(f_i).
\end{aligned}$$

The general case now follows easily by induction on the dimension of the wedge summands.

If X is a wedge sum of orbits, the suspension of X is a wedge of equivariant 1-spheres. By the wedge of circles axiom and Proposition 2.2 (iii), the claim holds for X .

For the induction step, let all wedge summands of X be at most n -dimensional. By the cofibration Axiom,

$$\ell_G(f) = \ell_G(f_{n-1}) + \ell_G(\hat{f}),$$

where f_{n-1} is the restriction of f to the $n-1$ -skeleton, where we may assume that f is cellular, and \hat{f} is the map induced on X_n/X_{n-1} . By induction, the claim holds for f_{n-1} and it holds for \hat{f} by what we have proven already. Hence, it holds for f itself. \square

Next we prove that Axioms 4a and 4b are equivalent.

Proposition 2.5. *An equivariant Lefschetz number satisfies the linearity axiom if and only if it satisfies the wedge of circles Axiom.*

Proof. We prove that linearity is equivalent to additivity in the sense of Proposition 2.4. By means of that Proposition, the result will follow.

We assume first that ℓ_G is linear. By induction, it then suffices to assume that $f : X \rightarrow X$ is a G -map of a complex consisting of two wedge summands, $X = A \vee B$.

Let $p_A : X \rightarrow X$ be the projection onto A , similarly p_B . Then Σf is equivariantly homotopic to $\Sigma(f \circ p_A) + \Sigma(f \circ p_B)$. By Proposition 2.2 (iii), commutativity and the facts $p_A^2 = p_A$, $\Sigma p_A = p_{\Sigma A}$ and similarly for B , we obtain

$$\begin{aligned}
\ell_G(f) &= -\ell_G(\Sigma f) \\
&= -\ell_G(\Sigma(f \circ p_A)) - \ell_G(\Sigma(f \circ p_B)) \\
&= -\ell_G(\Sigma f \circ p_{\Sigma A}) - \ell_G(\Sigma f \circ p_{\Sigma B}) \\
&= -\ell_G(\Sigma f \circ p_{\Sigma A}^2) - \ell_G(\Sigma f \circ p_{\Sigma B}^2) \\
&= -\ell_G(p_{\Sigma A} \circ \Sigma f \circ p_{\Sigma A}) - \ell_G(p_{\Sigma B} \circ \Sigma f \circ p_{\Sigma B}).
\end{aligned}$$

The map $p_{\Sigma A} \circ \Sigma f \circ p_{\Sigma A}$ leaves ΣA invariant and is the constant map on ΣB . Cofibration shows that

$$\ell_G(p_{\Sigma A} \circ \Sigma f \circ p_{\Sigma A}) = \ell_G(\Sigma f_A),$$

where $f_A : A \rightarrow A$ is the map induced by f via inclusion and projection. Of course a similar result holds with A replaced by B and consequently,

$$\ell_G(f) = -\ell_G(\Sigma f_A) - \ell_G(\Sigma f_B) = \ell_G(f_A) + \ell_G(f_B).$$

For the other direction, we assume that ℓ_G is additive. Let $f, h : \Sigma X \rightarrow \Sigma X$ be two G -maps. Let $\gamma : \Sigma X \rightarrow \Sigma X \vee \Sigma X$ be the folding map. By additivity,

$$\ell_G(\gamma \circ (f \vee h)) = \ell_G(p_1 \circ \gamma \circ f) + \ell_G(p_2 \circ \gamma \circ h),$$

where $p_i : \Sigma X \vee \Sigma X \rightarrow \Sigma X$ is the projection to the i -th summand, $i = 1, 2$. The maps $p_i \circ \gamma$ are equivariantly homotopic to the identity. By commutativity, we can conclude

$$\begin{aligned} \ell_G(f) + \ell_G(h) &= \ell_G(\gamma \circ (f \vee h)) \\ &= \ell_G((f \vee h) \circ \gamma) \\ &= \ell_G(f + h). \end{aligned}$$

□

With the additional linearity Axiom, there is a universal linear Lefschetz group U_GL and a universal linear equivariant Lefschetz number satisfying all four axioms. This is easily derived in the same manner as Proposition 2.1, with adding the linearity condition as an additional set of relations. By universality, there is a homomorphism $\mathbb{U}_GL \rightarrow U_GL$ such that the image of the universal equivariant Lefschetz number is the universal linear equivariant Lefschetz number. By abuse of notation, we will denote the universal linear equivariant Lefschetz number by L_G as well.

Under the assumption that every orbit map $G/H \rightarrow G/H$ is a homeomorphism, Laitinen and Lück proved that the universal Lefschetz group for linear equivariant Lefschetz numbers is isomorphic to

$$\bigoplus_{(H)} \mathbb{Z}Co(\pi_0(W(H))).$$

In the sum, (H) runs through the conjugacy classes of closed subgroups of G , $Co(K)$ for a group K denotes the set of conjugacy classes of elements of K and $W(H)$ is the Weyl group of H , that is, the quotient group $N(H)/H$, where $N(H)$ is the largest subgroup of G such that $H \subseteq N(H)$ is normal. In particular, the result implies that any linear equivariant Lefschetz number is uniquely determined by its values on self maps of orbits. Even more precisely, it is determined by its values on the maps

$$r_w : G/H_+ \rightarrow G/H_+, [g] \mapsto [gw],$$

where H runs through the conjugacy classes of closed subgroups of G and w , for each (H) , runs through a complete set of representants of $Co(\pi_0(W(H)))$.

We will reprove this result with slightly different techniques. In particular we will prove first that the universal Lefschetz group U_GL , without the additional linearity Axiom, is generated by representants of self-maps of finite wedge sums of equivariant 1-spheres, which already yields a useful normalization Axiom. Assuming linearity in addition, we show that for any topological group G , the basis of Lück and Laitinen at least generates the universal linear Lefschetz group, provided one replaces the Weyl group $W(H)$ by the space $(G/H)^H$. In most examples these two spaces are homeomorphic as topological monoids, for example if G is compact. When this is the case, we show that the set of generators is in fact a basis.

Theorem 2.6. *Let $A \subseteq \mathbb{U}_GL$ be the set of elements $[f, X] \in \mathbb{U}_GL$, where X is a finite wedge sum of equivariant 1-spheres. Let A' be the set of elements*

$[r_w, G/H_+] \in U_GL$, with H running through a complete set of representants of conjugacy classes of closed subgroups of G and r_w is right multiplication with w , w running through a complete set of representants of elements in $Co(\pi_0((G/H)^H))$. Here, conjugacy is to be understood via the $W(H)$ -action on $(G/H)^H$.

Then the universal Lefschetz group U_GL is generated by A and the universal linear Lefschetz group U_GL is generated by A' . If $W(H) \cong (G/H)^H$ for every closed subgroup of G , then A' is a basis of U_GL . More precisely, there are equivariant Lefschetz numbers ℓ_H with values in $\mathbb{Z}Co(\pi_0(W(H)))$ such that

$$U_GL \rightarrow \bigoplus_{(H)} \mathbb{Z}Co(\pi_0(W(H))), [f, X] \mapsto \sum_{(H)} \ell_H(f)$$

is an isomorphism.

Proof. We first prove that U_GL is generated by A . Let $f : X \rightarrow X$ be a self map of a finite G -CW complex of dimension n . We can assume that f is cellular and so

$$L_G(f) = L_G(f_{n-1}) + L_G(\hat{f}),$$

where \hat{f} is the map induced in the quotient $X_n/X_{n-1} \cong \bigvee_{i \in I} G/H_{i+} \wedge \mathbb{S}^n$. By Lemma 2.3, \hat{f} is equivariantly homotopic to the suspension of a G -map

$$h : \bigvee_{i \in I} G/H_{i+} \wedge \mathbb{S}^1 \rightarrow \bigvee_{i \in I} G/H_{i+} \wedge \mathbb{S}^1,$$

provided $n \geq 1$.

Therefore, $L_G(\hat{f}) = L_G(\Sigma^{n-1}h) = (-1)^{n-1}L_G(h)$. This, together with the formula $L_G(f) = L_G(f_{n-1}) + L_G(\hat{f})$, yields by induction that

$$L_G(f) = L_G(f_0) + R,$$

where R is a linear combination of elements in A . The suspension of a self map of a 0-dimensional G -CW complex is an element of A and

$$L_G(f) = -L_G(\Sigma f).$$

Hence, $L_G(f)$ is a linear combination of elements of A , which proves the first assertion.

Next we assume that we are dealing with the universal linear Lefschetz group U_GL . Obviously all the considerations above are still valid. Moreover by additivity, for a G -map

$$f : \bigvee G/H_{i+} \wedge \mathbb{S}^1 \rightarrow \bigvee G/H_{i+} \wedge \mathbb{S}^1,$$

$L_G(f) = \sum L_G(f_i)$, where the f_i are defined as before. This shows that U_GL is already generated by G -self maps of equivariant 1-spheres. By the usual reduction techniques,

$$[G/H_+ \wedge \mathbb{S}^1, G/H_+ \wedge \mathbb{S}^1]_G \cong \pi_1((G/H)_+^H \wedge \mathbb{S}^1) \cong \bigoplus_{w \in \pi_0(G/H^H)} \mathbb{Z}.$$

In particular, the maps $[[g], z] \mapsto [[gw], z]$ generate $[G/H_+ \wedge \mathbb{S}^1, G/H_+ \wedge \mathbb{S}^1]_G$, where w runs through representants of elements of $\pi_0((G/H)^H)$. These maps are the suspensions of maps r_w . Linearity again yields that U_GL is generated by the classes of maps r_w before passing to conjugacy classes. Finally by commutativity, $L_G(r_{v w v^{-1}}) = L_G(r_w)$ for $v \in W(H)$, which shows that we can pass to conjugacy classes, so A' indeed generates U_GL .

We now assume that $W(H) \cong (G/H)^H$ for every closed subgroup H of G . To show that the elements of A' are linearly independent, we need equivariant Lefschetz numbers for comparison purposes. Fix a closed subgroup H of G and let $f : X \rightarrow X$ be a G -self map of a finite G -CW complex X . The cellular chain complex $C_*(X^H, X^{>H})$ is a $\mathbb{Z}W(H)$ -module, and by homotopy invariance, it is a $\mathbb{Z}\pi_0(W(H))$ -module. Since $X^H \setminus X^{>H}$ is a free $W(H)$ -space, $C_*(X^H, X^{>H})$ is a free $\mathbb{Z}\pi_0(W(H))$ -module. We can assume that f is cellular and thus, f^H induces a $\mathbb{Z}\pi_0(W(H))$ -module homomorphism

$$C_*(f^H) : C_*(X^H, X^{>H}) \rightarrow C_*(X^H, X^{>H}).$$

$C_*(f^H)$ corresponds to a matrix with entries in $\mathbb{Z}\pi_0(W(H))$. The sum of the diagonal entries of this matrix followed by the projection to the abelianization of $\mathbb{Z}\pi_0(W(H))$ has all the properties of a trace map. The abelianization of $\mathbb{Z}\pi_0(W(H))$ is $\mathbb{Z}Co(\pi_0(W(H)))$. We obtain a trace map

$$\text{Tr} : \text{hom}_{\mathbb{Z}\pi_0(W(H))}(C_*(X^H, X^{>H}), C_*(X^H, X^{>H})) \rightarrow \mathbb{Z}Co(\pi_0(W(H))).$$

This is a special instance of the Hattori-Stallings trace, compare Section 4 and also [5].

With this trace map, we define

$$\ell_H(f) = \text{Tr } C_*(f^H).$$

It is easily seen that ℓ_H is a linear equivariant Lefschetz number. Hence, there is a homomorphism

$$U_GL \rightarrow \mathbb{Z}Co(\pi_0(W(H))), [f, X] \mapsto \ell_H(f).$$

We turn back to the question of linear independence of A' . Let an equation

$$\sum_{i \in I} \lambda_i a_i = 0$$

be given, where the a_i are pairwise distinct elements of A' . Let K be a closed subgroup of G with the property that there is an index $i \in I$ with $\lambda_i \neq 0$ and K is maximal with this property with respect to the partial order on isotropy types (H) induced by subconjugacy. Since

$$(G/H_+)^K = *$$

for subgroups H with either $(H) < (K)$ or with (H) and (K) incomparable, $\ell_K(a_i) = 0$ for elements a_i which are self maps of 1-spheres with type such an (H) . Thus, application of ℓ_K to the equation yields

$$\sum_{j \in J} \lambda_j \ell_K(a_j) = 0,$$

where $J \subseteq I$ is the set of all indices such that a_j is represented by a self map of G/K_+ . This complex consists of two 0-cells one of which is the base point, hence the cellular chain complex $C_*((G/K_+)^K, *)$ is isomorphic to $\mathbb{Z}\pi_0(W(K))$. The map r_w for $w \in W(K)$ induces the map

$$\mathbb{Z}\pi_0(W(K)) \rightarrow \mathbb{Z}\pi_0(W(K)), v \mapsto vw.$$

Therefore, its trace is $[w] \in \mathbb{Z}Co(\pi_0(W(K)))$. We conclude that

$$0 = \sum_{j \in J} \lambda_j \ell_K(a_j) = \sum_{j \in J} \lambda_j [w_j],$$

where $[w_j]$ is the unique class in $Co(\pi_0(W(K)))$ representing a_j . By definition of A' , the elements $[w_j]$ are linearly independent and so we must have $\lambda_j = 0$ for all $j \in J$, in contradiction to the choice of K . It follows that $\lambda_i = 0$ for all $i \in I$.

The statement concerning the explicit isomorphism follows along the lines of this proof. \square

Corollary 2.7. *An equivariant Lefschetz number is uniquely characterized by its values on G -self maps of finite wedge sums of equivariant 1-spheres. In particular, if (L_G^1, B) is an equivariant Lefschetz number and L_G^2 is an assignment of an element $L_G^2(f) \in B$ to every equivariant self map $f : X \rightarrow X$ of a finite G -CW complex, then $L_G^1(f) = L_G^2(f)$ for every self map f if and only if L_G^2 satisfies the following axioms.*

- (1) If $f : X \rightarrow X$ is equivariantly homotopic to $h : X \rightarrow X$, then $L_G^2(f) = L_G^2(h)$.
- (2) If $f : X \rightarrow Y$, $h : Y \rightarrow X$, then

$$L_G^2(f \circ h) = L_G^2(h \circ f).$$

- (3) If $A \subseteq X$ is a subcomplex and the diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ \downarrow f|_A & & \downarrow f & & \downarrow \hat{f} \\ A & \longrightarrow & X & \longrightarrow & X/A \end{array}$$

commutes, then

$$L_G^2(f) = L_G^2(f|_A) + L_G^2(\hat{f}).$$

- (4) If $X = \bigvee G/H_{i+} \wedge \mathbb{S}^1$ is a finite wedge sum and $f : X \rightarrow X$ is a G -map, then

$$L_G^2(f) = L_G^1(f).$$

Proof. The first three axioms imply that L_G^2 is an equivariant Lefschetz number. The fact that $L_G^2(f) = L_G^2(h)$ in a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow k & & \downarrow k \\ Y & \xrightarrow{h} & Y \end{array}$$

with a G -homotopy equivalence $k : X \rightarrow Y$ follows from Axioms 1. and 2. since

$$L_G^2(f) = L_G^2(k^{-1} \circ h \circ k) = L_G^2(k^{-1} \circ k \circ h) = L_G^2(h).$$

We conclude that L_G^2 is a homomorphic image of the universal Lefschetz number. Axiom 4. specifies the values of this homomorphism on generators of the universal Lefschetz group and consequently, L_G^2 and L_G^1 are induced by the same homomorphism, therefore equal. \square

The following corollary is immediate from the characterization of generators of the universal linear Lefschetz group.

Corollary 2.8. *If L^1 and L^2 are both linear, then Axiom 4 can be replaced by*

4b. For every closed subgroup H of G , $w \in Co(\pi_0((G/H)^H))$ and $r_w : G/H_+ \rightarrow G/H_+$, $[g] \mapsto [gw]$,

$$L_G^2(r_w) = L_G^1(r_w).$$

Finally, by the identification of a basis of the universal Lefschetz group, we have the following result.

Corollary 2.9. *If G is a topological group such that $W(H) \cong (G/H)^H$ for all closed subgroups H of G , then the set of equivariant Lefschetz numbers with values in a group B is in bijective correspondence to the set of set maps*

$$\bigcup_{(H)} Co(\pi_0(W(H))) \rightarrow B.$$

The correspondence is given by $\varphi \mapsto \ell_\varphi$, where ℓ_φ is the image of the universal linear Lefschetz number under the homomorphism

$$U_GL \rightarrow B, [r_w, G/H_+] \mapsto \varphi([w]).$$

3. EQUIVARIANT CELLULAR HOMOLOGY

In this section we briefly sketch the construction of equivariant cellular homology. We mainly follow [2] and [11]. Recall that a G -CW complex is a G -space X that is the colimit of its n -skeleta X_n . X_0 is a disjoint union of G -orbits G/H and X_{n+1} is constructed from X_n by taking the cofibre of a map from a disjoint union of equivariant spheres to X_n , i.e. a map

$$f : \coprod_{i \in I} G/H_i \times \mathbb{S}^n \rightarrow X_n.$$

The restriction of f to a summand is called an attaching map. X is said to be finite, if $X = X_n$ for some n and only finitely many spheres are attached at each stage. A based G -CW complex is a G -CW complex with a G -fixed base point. We do not assume attaching maps to be based.

We are going to describe a chain complex associated with the filtration of X by its skeleta that is analogous to the cellular chain complex of an ordinary CW complex. For missing details, we refer to [11]. The basic idea is to replace the summand \mathbb{Z} in the cellular complex corresponding to a cell by the functor $\mathbb{Z}[\bullet, G/H]_G$, corresponding to an equivariant cell of type H . In the following, \mathcal{O} will denote a full subcategory of the category of G -orbits and G -homotopy classes, consisting of finitely many objects. \mathcal{A} will denote the category of abelian groups.

Definition 2. Let X be a G -CW complex. We define the n -th cellular chain functor to be the functor

$$C_n^G(X) : \mathcal{O} \rightarrow \mathcal{A}, C_n^G(X)(G/K) = \bigoplus_H \bigoplus_{\substack{n\text{-cells} \\ \text{of type } H}} \mathbb{Z}[G/K, G/H]_G.$$

For a G -map $\varphi : G/K \rightarrow G/L$, $C_n^G(X)(\varphi)$ is defined as post-composition with φ on generators.

The functors $C_*^G(X)$ constitute a chain complex of contravariant functors $\mathcal{O} \rightarrow \mathcal{A}$. The natural transformation $d : C_n^G(X) \rightarrow C_{n-1}^G(X)$ is defined as follows. Let SX be the unreduced suspension of a G -space. That is, $SX = X \times [0, 1]/\sim$, where $X \times \{0\}$ is identified to a point and $X \times \{1\}$ is identified to another point. We

regard SX as a based G -space with base point $X \times \{1\}$. Let σ be an n -cell of X , $\tau \cong G/K_+ \wedge \mathbb{S}^{n-1}$ an $(n-1)$ -cell. Let $\alpha : G/H \times \mathbb{S}^{n-1} \rightarrow X_{n-1}$ be the attaching map of σ . α induces a map

$$G/H \times S\mathbb{S}^{n-1} \rightarrow SX_{n-1}, ([g], [x, t]) \mapsto [\alpha([g], x), t].$$

Since all points $([g], [x, t])$ with $t = 1$ are mapped to the base point of SX_{n-1} , this in turn induces a map

$$G/H_+ \wedge \mathbb{S}^n \rightarrow SX_{n-1},$$

which we can follow with the quotient map to

$$SX_{n-1}/SX_{n-2} \cong \Sigma(X_{n-1}/X_{n-2}).$$

Together with the projection p_τ to τ , we obtain a G -map

$$k : G/H_+ \wedge \mathbb{S}^n \rightarrow \Sigma(X_{n-1}/X_{n-2}) \xrightarrow{\Sigma p_\tau} G/K_+ \wedge \mathbb{S}^n.$$

k induces a map $\mathbb{S}^n \rightarrow \mathbb{S}^n$ in the group quotient. Let $-1 \in \mathbb{S}^{n-1} \subseteq S\mathbb{S}^n$, embedded as $\mathbb{S}^{n-1} \times \{\frac{1}{2}\}$. We find an equivariant homotopy of k such that the induced map is transverse to $-1 \in \mathbb{S}^n$. This implies that the preimage of -1 consists of finitely many points $x_1, \dots, x_{m_{\sigma, \tau}} \in \mathbb{S}^n$. Assuming that k itself has this property, we obtain that the preimage of $G/K \times \{-1\}$ under k is $G/H \times \{x_1, \dots, x_{m_{\sigma, \tau}}\}$. We define

$$\psi_j^{\sigma, \tau} : G/H \rightarrow G/K, \psi_j([g]) = k([g], x_j),$$

where we identify the point $k([g], x_j) = ([\tilde{g}], -1)$ with its first component. Furthermore, let $d_j^{\sigma, \tau}$ be the local degree of the map induced by k at x_j . Let $\varphi : G/L \rightarrow G/H$ be any G -map, then

$$d([\varphi] \cdot \sigma) = \sum_{\tau} \sum_{j=1}^{m_{\sigma, \tau}} d_j^{\sigma, \tau} [\psi_j^{\sigma, \tau} \circ \varphi] \cdot \tau,$$

and we extend linearly to all of $C_n^G(X)(G/H)$. It is clear that d constitutes a natural transformation. The fact that $d^2 = 0$ follows from the fact that forgetting the orbit maps yields the cellular chain functor of the quotient space X/G .

If $M : \mathcal{O} \rightarrow \mathcal{A}$ is any covariant functor, we obtain the equivariant chain complex with coefficients in M as

$$C_*^G(X; M) = C_*^G(X) \otimes_{\mathcal{O}} M.$$

The differential is given by $d \otimes \text{id}$. The homology of this chain complex is denoted by $H_n^G(X; M)$ and is called the n -th equivariant cellular homology of X . The exact definition of the tensor product $\otimes_{\mathcal{O}}$ is not important here, because we will find a more convenient description of $C_*^G(X, M)$ later.

To turn cellular homology into a functor, we have to specify C_n^G as a functor from the category of G -CW complexes to the category of contravariant functors $\mathcal{O} \rightarrow \mathcal{A}$. Thus, we have to define a natural transformation $C_n^G(f) : C_n^G(X) \rightarrow C_n^G(Y)$ for a G -map $f : X \rightarrow Y$. It is defined similarly to the boundary operator. Let $\iota_i : G/H_{i+} \wedge \mathbb{S}^n \rightarrow X_n/X_{n-1}$ be the inclusion of the i -th cell of X_n and let $p_j : Y_n/Y_{n-1} \rightarrow G/K_{j+} \wedge \mathbb{S}^n$ be the projection onto the j -th cell of Y . We can assume that f is cellular. Then we obtain a map

$$k : G/H_{i+} \wedge \mathbb{S}^n \xrightarrow{\iota_i} X_n/X_{n-1} \xrightarrow{f} Y_n/Y_{n-1} \xrightarrow{p_j} G/K_{j+} \wedge \mathbb{S}^n.$$

As above, we can assume that k induces a map \tilde{k} transverse to -1 in the group quotient and thus, k induces finitely many G -maps

$$\psi_\ell^{\sigma, \tau} : G/H_i \rightarrow G/K_j, \ell = 1, \dots, m.$$

Let $d_\ell^{\sigma, \tau}$ be the local degree of the map \tilde{k} at x_j and $\varphi : G/L \rightarrow G/H_i$ any G -map. Then we define

$$C_n^G(f)([\varphi] \cdot \sigma) = \sum_{\tau} \sum_{\ell=1}^{m_{\sigma, \tau}} d_\ell^{\sigma, \tau} [\psi_\ell^{\sigma, \tau} \circ \varphi] \cdot \tau$$

and extend linearly. Note again that by forgetting the orbit map part, we end up with the map f induces in the cellular homology of the quotient.

For a coefficient system M , the map $C_*^G(f; M)$ is defined as $C_*^G(f) \otimes \text{id}_M$.

We will now specify the coefficients we will use for cellular homology. Let \mathcal{I} be the free abelian group generated by all homotopy classes of G -maps between orbits. There is an obvious ring structure on \mathcal{I} , given by composition of homotopy classes whenever composition is possible, and being 0 otherwise. \mathcal{I} is called the isotropy ring associated to G (or \mathcal{O}). We can turn \mathcal{I} into a coefficient system as follows.

The value of \mathcal{I} on G/H is the ideal $\text{id}_{G/H} \cdot \mathcal{I}$. For a G -map $\varphi : G/H \rightarrow G/K$, $\mathcal{I}(\varphi)$ is pre-composition with φ . Since $\text{id}_{G/H}$ is an idempotent element in \mathcal{I} , $\mathcal{I}(G/H)$ is a projective right \mathcal{I} -module. Because of the isomorphism

$$C_*^G(X) \otimes_{\mathcal{O}} \mathcal{I} \cong \bigoplus_H \mathcal{I}(G/H)$$

stemming from the dual Yoneda isomorphism, the equivariant cellular chain complexes of X inherit the structure of projective right \mathcal{I} -modules as well.

4. TRACES

In order to be able to define the homological equivariant Lefschetz number, we need some background in the theory of traces for module endomorphisms of finitely generated projective modules over arbitrary rings. The general theory was developed by Hattori in [5], from which much of the following is taken. The resulting trace is called the Hattori-Stallings trace. We are especially interested in the behaviour of this trace with respect to homology.

In the following, R will denote an associative ring and by R -module we will mean a right R -module. For an R -module M , we let $DM = \text{hom}_R(M, R)$ be the dual module. Furthermore, we write $[M, N]$ for $\text{hom}_R(M, N)$, the R -module homomorphisms from M to N .

Definition 3. Let P be a finitely generated projective R -module. Let R^{ab} be the abelianization of R and let $\pi : R \rightarrow R^{ab}$ be the projection. The canonical map

$$\vartheta : P \otimes DP \rightarrow [P, P], \vartheta(x \otimes \xi)(y) = x \cdot \xi(y)$$

is an isomorphism and we can define

$$\text{Tr} : [P, P] \rightarrow R^{ab}, f \mapsto \pi \circ \varepsilon \circ \vartheta^{-1}(f),$$

where $\varepsilon : P \otimes DP \rightarrow R$ is the evaluation map. $\text{Tr}(f)$ is called the Hattori-Stallings trace of f .

Proposition 4.1. *The Hattori-Stallings trace is commutative in the following sense. If $f : P \rightarrow Q$, $g : Q \rightarrow P$ are module homomorphisms, then*

$$\text{Tr}(f \circ g) = \text{Tr}(g \circ f).$$

Proof. We define maps

$$\vartheta_{Q,P} : Q \otimes DP \rightarrow [P, Q], \quad \vartheta_{Q,P}(y \otimes \xi)(p) = y \cdot \xi(p).$$

Note that $\vartheta = \vartheta_{P,P}$. Furthermore, we define

$$\beta_{Q,P} : (Q \otimes DP) \times (P \otimes DQ) \rightarrow P \otimes DP, \quad (y \otimes \xi, x \otimes \eta) \mapsto y \cdot \xi(x) \otimes \eta.$$

Then the following diagram is easily seen to be commutative, where τ denotes the twisting map.

$$\begin{array}{ccccc}
 [Q, Q] & \xleftarrow{\circ} & [P, Q] \otimes [Q, P] & \xrightarrow{\tau} & [Q, P] \otimes [P, Q] & \xrightarrow{\circ} & [P, P] \\
 \uparrow \vartheta & & \uparrow \vartheta_{Q,P} \otimes \vartheta_{P,Q} & & \uparrow \vartheta_{P,Q} \otimes \vartheta_{Q,P} & & \uparrow \vartheta \\
 Q \otimes DQ & \xleftarrow{\beta_{P,Q}} & (Q \otimes DP) \otimes (P \otimes DQ) & \xrightarrow{\tau} & (P \otimes DQ) \otimes (Q \otimes DP) & \xrightarrow{\beta_{Q,P}} & Q \otimes DQ \\
 & \searrow \pi \circ \varepsilon & & & & & \swarrow \pi \circ \varepsilon \\
 & & R^{ab} & \xrightarrow{\text{id}} & R^{ab} & &
 \end{array}$$

The claim follows immediately. \square

Lemma 4.2. *Let P be a finitely generated projective module and $f : P \rightarrow P$ an endomorphism. If*

$$\vartheta^{-1}(\text{id}_P) = \sum x_i \otimes \xi_i \in P \otimes DP,$$

then

$$\vartheta^{-1}(f) = \sum f(x_i) \otimes \xi_i.$$

Proof. We calculate

$$\begin{aligned}
 \vartheta \left(\sum f(x_i) \otimes \xi_i \right) (y) &= \sum f(x_i) \cdot \xi_i(y) \\
 &= f \left(\sum x_i \cdot \xi_i(y) \right) \\
 &= f(y),
 \end{aligned}$$

and the claim follows. \square

Lemma 4.3. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of finitely generated projective R -modules and f a morphism of this exact sequence, then $\text{Tr } f_B = \text{Tr } f_A + \text{Tr } f_C$.*

Proof. Since all involved modules are projective, the exact sequence splits and we can view B as the direct sum $B = A \oplus C$. Let $i_A : A \rightarrow B$, $i_C : C \rightarrow B$ be the inclusions. Furthermore, we have the projections $p_A : B \rightarrow A$, $p_C : B \rightarrow C$ onto A and C , respectively, and these are idempotent elements in $[B, B]$, satisfying $p_A + p_C = \text{id}_B$. Consider the following four morphisms.

$$\begin{aligned}
 p_A \otimes Di_A : B \otimes DB &\rightarrow A \otimes DA, \quad x \otimes \xi \mapsto p_A(x) \otimes \xi \circ i_A, \\
 p_C \otimes Di_C : B \otimes DB &\rightarrow C \otimes DC, \quad y \otimes \eta \mapsto p_C(y) \otimes \eta \circ i_C, \\
 [B, B] &\rightarrow [A, A], \quad \varphi \mapsto p_A \circ \varphi \circ i_A, \\
 [B, B] &\rightarrow [C, C], \quad \varphi \mapsto p_C \circ \varphi \circ i_C.
 \end{aligned}$$

These constitute a commutative diagram

$$\begin{array}{ccc} B \otimes DB & \xrightarrow{p_A \otimes Di_A} & A \otimes DA \\ \vartheta_B \downarrow & & \downarrow \vartheta_A \\ [B, B] & \longrightarrow & [A, A] \end{array}$$

and similarly with A exchanged for C . Since id_B maps to id_A under the lower map, if we write

$$\vartheta_B^{-1}(\text{id}_B) = \sum x_i \otimes \xi_i,$$

we obtain

$$\vartheta_A^{-1}(\text{id}_A) = \sum p_A(x_i) \otimes \xi_i \circ i_A, \quad \vartheta_C^{-1}(\text{id}_C) = \sum p_C(x_i) \otimes \xi_i \circ i_C.$$

We have $i_A \circ f(a) = f(a)$ and $i_C \circ f(c) = f(c)$ for $a \in A, c \in C$. Thus we compute

$$\begin{aligned} \text{Tr } f_A + \text{Tr } f_B &= \pi \circ \varepsilon_A \circ \vartheta_A^{-1}(f_A) + \pi \circ \varepsilon_C \circ \vartheta_C^{-1}(f_C) \\ &= \pi \left(\sum \xi_i \circ i_A \circ f \circ p_A(x_i) \right) + \pi \left(\sum \xi_i \circ i_C \circ f \circ p_C(x_i) \right) \\ &= \pi \left(\sum \xi_i \circ f \circ p_A(x_i) \right) + \pi \left(\sum \xi_i \circ f \circ p_C(x_i) \right) \\ &= \pi \left(\sum \xi_i \circ f(p_A(x_i) + p_C(x_i)) \right) \\ &= \pi \left(\sum \xi_i \circ f(x_i) \right) \\ &= \pi \circ \varepsilon_B \circ \vartheta_B^{-1}(f) \\ &= \text{Tr } f. \end{aligned}$$

□

5. THE HOMOLOGICAL LEFSCHETZ NUMBER

Let \mathbb{U}_G be the tom Dieck ring of G . That is,

$$\mathbb{U}_G = \bigoplus_{(H)} \mathbb{Z} \cdot (H),$$

where the sum ranges over all conjugacy classes of closed subgroups of G . \mathbb{U}_G is the domain of the universal equivariant Euler characteristic, constructed similarly to the universal equivariant Lefschetz group, compare [10]. This will however not be of concern in the following.

Recall the definition of the isotropy ring \mathcal{I} from Section 3. There is an obvious augmentation map $\varepsilon : \mathcal{I} \rightarrow \mathbb{U}_G$, defined as follows. Let $\varphi : G/H \rightarrow G/K$ be a G -map. Then the basis element $[\varphi]$ of \mathcal{I} maps to $1 \cdot (H)$ if and only if $H = K$, and to 0 otherwise. With the help of this augmentation, we can define the topological equivariant Lefschetz number.

Definition 4. Let X be finite a G -CW complex, $f : X \rightarrow X$ a G -map. The unreduced homological equivariant Lefschetz number of f is defined to be

$$\Lambda_G(f) = \sum_{n=0}^{\infty} (-1)^n \cdot \varepsilon(\text{Tr } C_n^G(f)).$$

The (reduced) homological equivariant Lefschetz number is defined as

$$L_G(f) = \Lambda_G(f) - 1 \cdot (G).$$

It is not hard to see that the homological equivariant Lefschetz number is indeed a linear equivariant Lefschetz number in the sense of the definition.

Proposition 5.1. *The homological equivariant Lefschetz number satisfies the first three axioms of Corollary 2.7 and is linear.*

Proof. The homotopy axiom is obvious. The commutativity axiom follows from the commutativity of the trace, Proposition 4.1. Let X, Y be finite G -CW complexes and let $f : X \rightarrow Y$, $h : Y \rightarrow X$ be G -maps. Then

$$\begin{aligned} \Lambda_G(f \circ h) &= \sum_{n=0}^{\infty} (-1)^n \text{Tr} (C_n^G(f \circ h; \mathcal{I})) \\ &= \sum_{n=0}^{\infty} (-1)^n \text{Tr} (C_n^G(f; \mathcal{I}) \circ C_n^G(h; \mathcal{I})) \\ &= \sum_{n=0}^{\infty} (-1)^n \text{Tr} (C_n^G(f; \mathcal{I})) \cdot \text{Tr} (C_n^G(h; \mathcal{I})) \\ &= \sum_{n=0}^{\infty} (-1)^n \text{Tr} (C_n^G(h; \mathcal{I})) \cdot \text{Tr} (C_n^G(f; \mathcal{I})) \\ &= \Lambda_G(h \circ f). \end{aligned}$$

This clearly carries over to the reduced case.

For the cofibration axiom, consider a cofibre sequence $A \xrightarrow{i} X \xrightarrow{p} X/A$. We have the short exact sequences

$$0 \rightarrow C_n^G(A; \mathcal{I}) \rightarrow C_n^G(X; \mathcal{I}) \rightarrow C_n^G(X; \mathcal{I})/C_n^G(A; \mathcal{I}) \rightarrow 0,$$

and for $n \geq 1$, $C_n^G(X; \mathcal{I})/C_n^G(A; \mathcal{I})$ is isomorphic to $C_n^G(X/A; \mathcal{I})$. For $n = 0$,

$$C_0^G(X; \mathcal{I})/C_0^G(A; \mathcal{I}) = \bigoplus_{\sigma} \bigoplus_K \mathbb{Z}[G/K, G/H]_G,$$

where the first sum ranges over the 0-cells of type H which are not contained in A_0 . On the other hand,

$$C_0^G(X/A; \mathcal{I}) = C_0^G(X; \mathcal{I})/C_0^G(A; \mathcal{I}) \oplus \bigoplus_K \mathbb{Z}[G/K, G/G]_G,$$

the last summand representing the base point. The above short exact sequence for $n = 0$ yields

$$\text{Tr} (C_0^G(f)) = \text{Tr} (C_0^G(f|_A)) + \text{Tr} (\tilde{f}),$$

where \tilde{f} is the map $C_0^G(f)$ induces in the quotient. From the short exact sequence

$$0 \rightarrow C_0^G(*; \mathcal{I}) \rightarrow C_0^G(X/A; \mathcal{I}) \rightarrow C_0^G(X; \mathcal{I})/C_0^G(A; \mathcal{I}) \rightarrow 0,$$

we have $\text{Tr}(C_0^G(\hat{f})) = \text{Tr}(C_0^G(*)) + \text{Tr}(\tilde{f}) = 1 \cdot (G) + \text{Tr}(\tilde{f})$. Putting all this together yields

$$\begin{aligned} \Lambda_G(f) &= \sum_{n=0}^{\infty} (-1)^n \text{Tr}(C_n^G(f; \mathcal{I})) \\ &= \sum_{n=1}^{\infty} (-1)^n \text{Tr}(C_n^G(f|_A; \mathcal{I})) + \sum_{n=1}^{\infty} (-1)^n \text{Tr}(C_n^G(\hat{f}; \mathcal{I})) \\ &\quad + \text{Tr}(C_0^G(f; \mathcal{I})) + \text{Tr}(C_0^G(\hat{f}; \mathcal{I})) - 1 \cdot (G) \\ &= \Lambda_G(f|_A) + \Lambda_G(\hat{f}) - 1 \cdot (G). \end{aligned}$$

This proves the cofibration axiom for the reduced variant.

Finally for linearity, we prove that additivity holds on the wedge sum of two complexes. By Propositions 2.4 and 2.5, this is more than sufficient to prove the claim. We have

$$C_n^G(X \vee Y; \mathcal{I}) = C_n^G(X; \mathcal{I}) \oplus C_n^G(Y; \mathcal{I})$$

for $n > 0$. For $n = 0$, $C_0^G(X \vee Y; \mathcal{I})$ is the free abelian group with basis the sets $[G/K, G/H]_G$ for each 0-cell of X or Y of type H which is not the basepoint, together with basis elements $[G/K, G/G]_G$ corresponding to the common base point of X and Y . The canonical map $C_0(X; \mathcal{I}) \oplus C_0(Y; \mathcal{I}) \rightarrow C_0(X \vee Y; \mathcal{I})$ has kernel $C_0(*; \mathcal{I})$ and we obtain the exact sequence

$$0 \rightarrow C_0^G(*; \mathcal{I}) \rightarrow C_0^G(X; \mathcal{I}) \oplus C_0^G(Y; \mathcal{I}) \rightarrow C_0^G(X \vee Y; \mathcal{I}) \rightarrow 0.$$

Consequently, $\text{Tr} C_0^G(f) + \text{Tr} C_0^G(g) = \text{Tr} C_0^G(f_X) + \text{Tr} C_0^G(f_Y)$, where $* : * \rightarrow *$ is the unique G -map. Clearly, this map has trace $1 \cdot (G)$, and hence,

$$\begin{aligned} L_G(f) &= \Lambda_G(f) - 1 \cdot (G) \\ &= \sum_{n=0}^{\infty} (-1)^n \varepsilon(\text{Tr} C_n^G(f)) - 1 \cdot (G) \\ &= \sum_{n=1}^{\infty} (-1)^n (\varepsilon(\text{Tr} C_n^G(f_X)) + \varepsilon(\text{Tr} C_n^G(f_Y))) \\ &\quad + \varepsilon(\text{Tr} C_0^G(f_X)) + \varepsilon(\text{Tr} C_0^G(f_Y)) - 2 \cdot (G) \\ &= L_G(f_X) + L_G(f_Y). \end{aligned}$$

□

We proceed to calculate the value of L_G on finite wedge sums of equivariant 1-spheres in order to be able to compare it, by means of Corollary 2.7, to other equivariant Lefschetz numbers.

Proposition 5.2. *Let $X = \bigvee_{i \in I} G/H_{i+} \wedge \mathbb{S}^1$ be a finite wedge of equivariant 1-spheres and $f : X \rightarrow X$ a G -map. Let f_i be the map f induces on the i -th summand via inclusion and projection. Then*

$$L_G(f) = \sum_{i \in I} L(\bar{f}_i) \cdot (H),$$

where \bar{f}_i is the map f_i induces in the group quotient and L denotes the ordinary reduced Lefschetz number.

Proof. By linearity, Proposition 5.1, it suffices to assume that X is a single equivariant 1-sphere. We can assume for the rest of the proof that \bar{f} is transverse regular at $-1 \in \mathbb{S}^1$. The map induced by \bar{f} in first homology is multiplication by the degree of \bar{f} , which is the sum of local degrees, computed at the preimages of -1 . To write down a formula, let $\bar{f}^{-1}(-1) = \{x_1, \dots, x_m\}$ and d_j be the local degree of \bar{f} at x_j , $j = 1, \dots, m$, then

$$H_1(\bar{f})(\alpha) = \sum_{j=1}^m d_j \cdot \alpha.$$

It follows that the Lefschetz number of \bar{f} is $1 - \sum_{j=1}^m d_j$ and so for the reduced Lefschetz number we have $L(\bar{f}) = -\sum_{j=1}^m d_j$.

For the equivariant Lefschetz number, we note that

$$C_1^G(X; \mathcal{I}) = \bigoplus_K \mathbb{Z}[G/K, G/H]_G.$$

Let $\psi_j : G/H \rightarrow G/H$ be the G -map obtained by restricting f to $G/H \times \{x_j\}$, $j = 1, \dots, m$. By definition, the induced map in the first cellular chain module of X is given as

$$C_1^G(f; \mathcal{I})([\varphi]) = \sum_{j=1}^m d_j \cdot [\psi_j \circ \varphi],$$

where $\varphi : G/K \rightarrow G/H$ is any G -map. Now let

$$x = [\text{id}_{G/H}] \in C_1^G(X; \mathcal{I})$$

and

$$\xi : C_1^G(X; \mathcal{I}) \rightarrow \mathcal{I}, [\varphi] \mapsto [\varphi],$$

extended linearly. Since $[\varphi] = \xi(x \cdot [\varphi])$, we have that

$$\vartheta^{-1}(\text{id}_{C_1^G(X; \mathcal{I})}) = x \otimes \xi.$$

Thus, $\vartheta^{-1}(C_1^G(f; \mathcal{I})) = C_1^G(f; \mathcal{I})(x) \otimes \xi$ by Lemma 4.2, and the trace of $C_1^G(f; \mathcal{I})$ is calculated as

$$\text{Tr}(C_1^G(f; \mathcal{I})) = \xi(C_1^G(f; \mathcal{I})(x)) = \sum_{j=1}^m d_j [\psi_j] \in \mathcal{I}^{ab}.$$

The augmentation sends all elements $[\psi_j]$ to $1 \cdot (H)$, since these are self-maps of G/H . Therefore,

$$\varepsilon(\text{Tr}(C_1^G(f; \mathcal{I}))) = \sum_{j=1}^m d_j \cdot (H).$$

Clearly, $\varepsilon(\text{Tr}(C_0^G(f; \mathcal{I}))) = 1 \cdot (G)$, since we have a single 0-cell of type (G) . We conclude that

$$\Lambda_G(f) = 1 \cdot (G) - \sum_{j=1}^m d_j \cdot (H),$$

which proves that indeed $L_G(f) = L(\bar{f}) \cdot (H)$. \square

Corollary 5.3. *The homological equivariant Lefschetz number is the unique linear equivariant Lefschetz number L_G such that $L_G(\varphi) = 1 \cdot (H)$ for every self map φ of G/H_+ .*

Proof. The quotient of the suspension of a G -map of orbits is the identity on \mathbb{S}^1 , so its reduced non-equivariant Lefschetz number is -1 . The claim follows from Proposition 5.2 and Corollary 2.8. \square

6. THE ANALYTICAL LEFSCHETZ NUMBER

In this section we will define a homotopy invariant on G -ANRs that will be seen to satisfy the axioms of Corollary 2.7. It was originally constructed by Dzedzej in [3]. We briefly recall some facts from the theory of ANRs and G -ANRs.

Definition 5. An absolute neighbourhood retract, or ANR, is a metric space X satisfying the following universal property. Whenever $i : X \rightarrow M$ is an isometric embedding of X into a metric space M , there is a neighbourhood $U \subseteq M$ of $i(X)$ and a retraction $r : U \rightarrow X$.

The notion of a G -ANR is apparent. The basic theory of G -ANRs was developed by Murayama in [8]. There he proves the following proposition. Recall the notion of isotropy subgroup of an element x in a G -space X . It is the group $G_x = \{g \in G \mid g.x = x\}$. We denote

$$\begin{aligned} X_{(H)} &= \{x \in X \mid (G_x) = (H)\} \\ X_{\geq(H)} &= \{x \in X \mid (G_x) \geq (H)\} \\ X_{>(H)} &= \{x \in X_{\geq(H)} \mid (G_x) \neq (H)\}, \end{aligned}$$

the partial order, as before, being subconjugacy of conjugacy classes of closed subgroups.

Proposition 6.1. *Let H be a closed subgroup of G and let X be a G -ANR. Then the spaces $X_{(H)}$, $X_{\geq(H)}$ and $X_{>(H)}$ are G -ANRs and the quotient X/G is an ordinary ANR. Furthermore, any finite G -CW complex is a G -ANR.*

Let $X \subseteq U$ be a pair of ANRs, $f : X \rightarrow U$ a continuous map having no fixed points on ∂X . For simplicity we assume that U can be embedded into \mathbb{R}^n via an embedding $i : U \rightarrow \mathbb{R}^n$, otherwise we have to put some compactness assumption on f . Now i embeds X as well and so we find a neighbourhood V of $i(X)$ and a retraction $r : V \rightarrow i(X)$. The map $i \circ f \circ i^{-1} \circ r : V \rightarrow V$ is well-defined and has no fixed points outside of $i(X)$. Hence, its fixed point index (see [9]) is well-defined and we write $i(f, X, U)$ for this number. It is shown in [9] that this index is well-defined and enjoys all the usual properties of an index.

Next we take $X \subseteq U$ to be a pair of G -ANRs, $f : X \rightarrow U$ a G -map without fixed orbits on ∂X . Since X/G and U/G are ANRs, the fixed point index of the induced map $\bar{f} : X/G \rightarrow U/G$ is defined and so are the various indices of the induced maps $\bar{f}_{\geq(H)} : X_{\geq(H)}/G \rightarrow U_{\geq(H)}/G$ and $\bar{f}_{>(H)} : X_{>(H)}/G \rightarrow U_{>(H)}/G$ for closed subgroups $H \subseteq G$. So to each orbit type (H) of X , we can assign the integer

$$i_{(H)}(f, X, U) = i(\bar{f}_{\geq(H)}, X_{\geq(H)}/G, U_{\geq(H)}/G) - i(\bar{f}_{>(H)}, X_{>(H)}/G, U_{>(H)}/G).$$

Assume $X \subseteq U$ are G -ANRs with finite orbit type. Then the fixed orbit index of f is defined to be the element

$$i_G(f, X, U) = \sum_{(H)} i_{(H)}(f) \cdot (H) \in \mathbb{U}_G,$$

In particular, if $X = U$ is a finite G -CW complex, the fixed orbit index of a G -map $f : X \rightarrow X$ is defined. In that case we write $i_G(f, X, X)$ for $i_G(f, X, X)$.

We have augmentation maps

$$\varepsilon_{(H)} = U_G \rightarrow \mathbb{Z}, \quad \varepsilon_{(H)}(a) = \sum_{(K) \geq (H)} \pi_{(K)}(a),$$

where $\pi_{(K)} : U_G \rightarrow \mathbb{Z}$ is the projection onto the (K) -th summand. Thus, $\varepsilon(H)$ takes all the coefficients of orbit types less or equal to (H) and sums them up.

Definition 6. Let X be a G -CW complex and $f : X \rightarrow X$ a G -map. The reduced fixed orbit index of f is defined to be the element

$$L_G(f) = i_G(f, X) - 1 \cdot (G).$$

We note that in fact the reduced fixed orbit index can be defined as

$$L_G(f) = \sum_{(H)} \left(\hat{i}(\overline{f_{\geq(H)}}), X_{\geq(H)}/G - \hat{i}(\overline{f_{>(H)}}), X_{>(H)}/G \right) \cdot (H),$$

where \hat{i} is the reduced fixed orbit index, i.e. $\hat{i}(f) = i(f) - 1$. Obviously, all the terms coming from the -1 -part cancel out except for the last one corresponding to (G) , which shows that the two definitions are equal. Since the fixed point index equals the Lefschetz number on finite CW complexes, the last equality takes the more convenient form

$$L_G(f) = \sum_{(H)} \left(L(\overline{f_{\geq(H)}}) - L(\overline{f_{>(H)}}) \right) \cdot (H)$$

Proposition 6.2. *The reduced fixed orbit index satisfies the first three axioms of Corollary 2.7.*

Proof. This follows immediately from the fact that the non-equivariant Lefschetz number satisfies the corresponding axioms with trivial group action. \square

We compute the value of the fixed orbit index on a wedge sum of equivariant 1-spheres. Along the lines it turns out that the fixed orbit index is a linear equivariant Lefschetz number, which of course could also have been proven directly.

Proposition 6.3. *Let $X = \bigvee_{i \in I} G/H_{i+} \wedge \mathbb{S}^1$ be a finite wedge sum of equivariant 1-spheres and $f : X \rightarrow X$ a G -map. Then*

$$L_G(f) = \sum_{i \in I} L(\overline{f_i}) \cdot (H_i),$$

where f_i is the self map of $G/H_{i+} \wedge \mathbb{S}^1$ induced by f via inclusion of and projection to the i -th summand and $\overline{f_i}$ is the map f_i induces in the group quotient. In particular, the fixed orbit index equals the homological equivariant Lefschetz number by Corollary 2.7.

Proof. For any closed subgroup H of G , we have that $X_{>(H)}$ is an invariant subcomplex of $X_{\geq(H)}$ and their quotient is the wedge sum of those equivariant spheres $G/H_{i+} \wedge \mathbb{S}^1$ such that $(H_i) = (H)$. Passing to the group quotient and using the cofibration axiom for the ordinary reduced Lefschetz number, this yields

$$L(\overline{f_{\geq(H)}}) = L(\overline{f_{>(H)}}) + L(\overline{f_{(H)}}).$$

Inserting into the definition of the reduced fixed orbit index, we obtain

$$\begin{aligned} L_G(f) &= \sum_{(H)} \left(L(\bar{f}_{\geq(H)}) - L(\bar{f}_{>(H)}) \right) \cdot (H) \\ &= \sum_{(H)} L(\bar{f}_{(H)}) \cdot (H). \end{aligned}$$

Since the non-equivariant reduced Lefschetz number is additive on wedge sums, we have

$$L(\bar{f}_{(H)}) = \sum_i L(\bar{f}_i),$$

where the sum runs over all indices i such that $(H_i) = (H)$ and f_i is defined as above. This proves the claim. \square

7. THE LEFSCHETZ FIXED ORBIT THEOREM

The theorem is easily deduced from the analytical definition of the Lefschetz number. A topological proof is also possible.

Theorem 7.1. *Let X be a finite G -CW complex and $f : X \rightarrow X$ a G -map. Let $\pi_{(H)} : U_G \rightarrow \mathbb{Z}$ be the projection to the (H) -component of U_G . If $(H) \neq G$ and $\pi_{(H)}(L_G(f)) \neq 0$, f has a non-trivial fixed orbit of orbit type at least (H) . If the base point $* \in X$ is an isolated point, then the results holds for $(H) = (G)$ as well.*

Proof. By definition of the analytical equivariant Lefschetz number,

$$\pi_{(H)}(L_G(f)) = i(\overline{f_{\geq(H)}}, X_{\geq(H)}/G) - i(\overline{f_{>(H)}}, X_{>(H)}/G)$$

for $(H) \neq (G)$ and

$$\pi_{(G)}(L_G(f)) = i(\overline{f_{\geq(G)}}, X_{(G)}/G) - 1.$$

If the former is non-zero, then either

$$i(\overline{f_{\geq(H)}}, X_{\geq(H)}/G)$$

or

$$i(\overline{f_{>(H)}}, X_{>(H)}/G)$$

will be non-zero. Both facts imply that \bar{f} has a fixed point in the set of points with orbit type at least (H) , and this implies existence of a fixed orbit of f of type at least (H) . Non-triviality follows since if the base point were the only fixed orbit, we would have

$$i(\overline{f_{\geq(H)}}, X_{\geq(H)}/G) = i(\overline{f_{>(H)}}, X_{>(H)}/G).$$

Under the additional assumption that the base point is isolated, non-vanishing of $i(\overline{f_{\geq(G)}}, X_{(G)}/G) - 1$ implies that there must be at least one more fixed point of f in $X_{(G)}$, apart from the base point, since $i(\overline{f_{\geq(G)}}, X_{(G)}/G)$ is different from 1 and the local index of the base point is 1. \square

We give a simple example that shows that the term 'at least' can not be deleted in the statement of the theorem. Consider the \mathbb{Z}_2 -complex $\mathbb{S}^1 \subseteq \mathbb{C}$, with the action given by reflection at the imaginary axis. The quotient space can be identified with $[-1, 1]$ in the obvious way. Any continuous map $[-1, 1] \rightarrow [-1, 1]$ fixing -1 and 1 lifts to an equivariant self map of \mathbb{S}^1 . For example, we can consider the map

$$f_\mu : [-1, 1] \rightarrow [-1, 1], \quad f_\mu(t) = t^3 - \mu t^2 + \mu$$

for $\mu \in [0, 1]$. f induces (not uniquely) an equivariant homotopy F of G -self maps of \mathbb{S}^1 . By definition of the fixed orbit index, we have to calculate the Lefschetz number of f_μ in order to obtain the equivariant Lefschetz number of F_μ . A simple calculation yields that f_0 has Lefschetz number 1. The restriction to $\{-1, 1\}$, which is the quotient of the \mathbb{Z}_2 -fixed space of \mathbb{S}^1 , is the identity on a finite set, hence its Lefschetz number is 2. We obtain that

$$i_G(F_\mu, \mathbb{S}^1) = -(e) + 2 \cdot (\mathbb{Z}_2),$$

and therefore,

$$L_G(F_\mu) = -(e) + (\mathbb{Z}_2).$$

In particular, the (e) -component of $L_G(F_\mu)$ is non-zero. But F_1 has no fixed orbits apart from the trivial ones, in particular, it has no fixed orbit of type (e) .

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